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ABSTRACT

Working in groups of three, seventh-grade students were video taped solving an applied mathematical problem. Audio tapes were made of each group discussing their session. Analysis focused on the first stage of Noddings (1984) model of problem solving for school mathematics: Creation of a representation. Some factors identified as impeding formation of a useful representation were: (a) lack of experience; (b) imposing unrequired restrictions on the problem; (c) lack of metacognitive skills; and (d) the influence of beliefs. Group interaction frequently offset these factors. (Author)

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Factors Impeding the Formation of a Useful Representation in Mathematical Problem Solving

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Abstract

Working in groups of three, seventh-grade students were video taped solving an applied mathematical problem. Audio tapes were made of each group discussing their session. Analysis focused on the first stage of Noddings (1984) model of problem solving for school mathematics: Creation of a representation. Some factors identified as impeding formation of a useful representation were: (a) lack of experience, (b) imposing unrequired restrictions on the problem, (c) lack of metacognitive skills, and (d) the influence of beliefs. Group interaction frequently offset these factors.

The analysis in this report is based on video and audio transcipts of 12 average ability seventh-grade students working in groups of three solving an applied mathematical problem. The problem-solving process was traced using Noddings (1984) model of problem solving for school mathematics:

(1) creation of a representation;

(2) executing a plan based on the representation;

(3) undergoing the consequences;

and, (4) evaluating the results.

Students whose solution path was based on an inaccurate representation of the problem situation (Step 1) were generally not successful in solving the problem.

Some factors that affected individual ability to form an accurate representation of the problem were identified. Working in a small group offset some of these factors and frequently enhanced problem-solving success. The subjects employed heuristic behaviors frequently during their sessions. However, when implemented during a solution path that was based on an inaccurate problem representation, they proved of little consequence in overall problem-solving success.

Theoretical Perspective

In her model for school problem solving Noddings (1984) argues that the first stage in problem solving is to create a picture of the situation. It is at this stage that "the mind is engaged, the intuitive capacity is actively looking, and all sorts of things happen . . . If we want seriously to recognize the role of intuition in problem solving--and virtually all mathematicians recognize its contribution--then we need to fill out this stage of representation with concrete accounts rather than detailed and prespecified theoretical steps" (p. 8). Since creating a representation is not an observable action a method or methods must be devised to make the internalized dialogue of the problem solver open to investigation.

A common method to address this problem in process-variable research has been the single student protocol or "think aloud" method which asks the problem solver to verbalize his or her thoughts while solving the problem (Gilbert & Lietz, 1982). Ginsburg (1981) suggests that one of the main purposes of this method is the discovery of cognitive activity. Noddings (1982) argues that this method may channel the subjects into areas they might not normally have thought of and that it may have a constraining effect on the use of highly inventive heuristics.

An alternate approach employed in this study is the use of a small group. Although the group has long been studied outside the field of mathematics .

education, Silverman (1972) points out that much previous research has studied the group as a social system and has addressed questions of how people behave in a group and how group products compare to individuals' products. The investigation of individuals working in a group is a relatively untapped resource in mathematical problem sovlving. Silver (1985) suggests that small groups may "promote this externalized, think aloud protocol more readily than individual interviews, and they may provide a useful vehicle for studying such processes (often covert in individual protocols) as planning, critiquing, monitoring and evaluating" (p. 33).

Methods

Subjects

From the seventh grade of a middle-class urban elementary school, 12 average-ability subjects were identified. Average mathematics achievement was determined by three factors: scoring within one grade level in both overall mathematics and reading on the California Achievement Test, being on grade-level in a continuous progress program in both reading and mathematics, and informal teacher input on overall classroom performance. The subjects were placed in three groups of two girls and one boy and one group of two boys and one girl.

Task

The problem used in this study was a four-part applied problem developed by Lesh (1982) in his Applied Problem Solving Project.

Carpentry Problem

John is constructing a recreation room in his basement. He has put up the walls and put down the floor. He needs to buy baseboard to put along the walls. The baseboard comes in 10-foot and 16 foot lengths. How many boards of each kind should he buy?

If John wants to have as few seams as possible, how many of each size should he buy?

If John wants to have as little waste as possible, how many of each size should he buy?

If the 16-foot boards cost $1.25 per foot and the 10-foot boards cost $1.10 per foot, how many of each kind should he buy to spend the least amount of money?

The subjects were given rulers, pencils, papers and calculators to aid in solving the problem. Each problem part was presented on a new sheet of paper. The students were instructed to turn over one sheet of paper at a time. They could refer back to any work they had done on previous parts, but were not to look ahead at new parts.

Follow Up

A follow up discussion with each group was audio taped. Hypotheses raised during observation of the video tapes were pursued with the group to clarify formation of themes and patterns.

Discussion

Heuristic Behaviors

None of the groups was able to solve all four parts of the Carpentry Problem. Although they exhibited frequent instances of heuristic behaviors, such as drawing a diagram, rereading the problem, thinking of a specific case, restating the problem, and trial and error, these heuristics were generally not helpful when they had not created a useful representation of the problem. Consider the following sequence.

1.	Christy	(Reading) If John wants to have as few seams . . . O.K. . . . If John wants to have as few seams as possible, how many of each size should he buy?	
2.	Darby	We already answered that. (Reads problem silently.) We already answered that in the last problem.	Reread the problem & Think of a similar problem
3.	Leigh	No we didn't.	
4.	Christy	I don't think we did. (Rereads the problem) . . . the base of it, the size of the base.	Reread the problem & Restate the problem
5.	Darby	The base?	
6.	Christy	(Calculating on paper) He needs 2.5' for the 10-foot boards and what was the other inch . . .16 foot . . .he needs one and one-half for the . . .	
7.	Darby	Can't you multiply these two and get the answer?	
8.	Leigh	I got it.	
9.	Darby	You always get it.	

10.	Christy	You're smarter that's what.	
11.	Leigh	I got two 16-inch boards and uh	
12.	Darby	Sixteen foot.	
13.	Leigh	O.K. and, uh, one 10-foot boards which is 16 times 2 equals 32 and 32 plus 10 equals 42.	
14.	Darby	The perimeter of the room is 98.	
15.	Leigh	No, but on this we are just doing the base, we are not doing the whole room. We are just doing the base. The base	
16.	Darby	(interrupts) but the perimeter.	
17.	Leigh	. . . which is 42. We are doing the base as this. (Points to diagram). See the boards. We are doing this right there.	Draws a diagram
18.	Darby	Forty-two feet? How many? He is going to buy 42 boards, right?	
19.	Christy	No, 42 bases!	Draws a diagram to show base of the room

20. Leigh One 10-foot and two 16-foot.

21. Darby Forty-two bases?

As is noted, the subjects in this short sequence employed several heuristics that did not aide them in their problem-solving process. In statement four Christy uses two very powerful heuristics; (a) reread the problem, and (b) restate the problem in your own words. Later in the discussion she draws a diagram (Figure 1) of her "base of the room" to share with Darby.

Figure 1. Christy's Base of the Room

None of the three strategies was useful to the group since they had not formed an accurate representation of what it meant to put baseboard around the edge of the room. In fact, the discussion continued on for several minutes. Darby finally succumbed to the pressure of other members of the group and invented an algorithm to find the invented base of the room.

Darby Why don't we do this? Take . . .
the room has four sides, right?
Take. . .that . . . now we want the
base of it . . just the base.
That's before we do all this, so we
take 1/4 divide by 98, then we
should see our base.

Think of a useful formula

Factors Inhibiting Creation of a Useful Representation

An attempt was then made to isolate categories of factors that hindered an individuals ability to the form a useful representation of the problem. The following were identified: (a) lack of experience; (b) imposing unrequired restrictions on the problem; (c) lack of metacognitive skills; and (e) the influence of belief systems.

Lack of Experience. When previous experience produced no connection for the subject to draw upon in forming a representation, the subject frequently invented definitions or even algorithms for the situation. These invented approaches were usually cued by some word or set of words in the problem that he or she associated with a mathematical concept.

The previous sequence with Darby, Leigh and Christy illustrated this point. They had reached a very reasonable solution to Part 1 of the problem, concluding that to cover the 98-foot perimeter of the room with baseboard they would need five 10-foot boards and three 16-foot boards. However, the introduction of a new situation in Part 2 for which they apparently had no experience resulted in the sequence above.

Another example occured with Ricky and Lisa from Group 1. After a direct question to the investigator, "Does seams mean empty space?" which was responded to by "No, it is the place where two boards meet" the group was still unable to form a representation of the problem. After some discussion, Damon engaged in the following reflective talk and finally directs a comment to Lisa.

Damon — No it's 1.8 . . . we've got to look this over a little bit . . . not really sure how much . . . no, cause if we did that with 16-foot boards . . . we want to make it as less as few seams as possible. This (pointing to 1.8) has to be a larger number. Try 10 . . . 3,10 . . . we could write it down 3 for the 10 inch boards . . . uh, for the 28 foot .

Lisa Three for the 10-inch boards?

Damon Change that cause we need as many boards as possible.

Even after the term "seams" had been defined for him, Damon did not have any experience to call upon to help him create a representation of the seams formed by the baseboards. His representation was that in order to have as few seams as possible you must have as many boards as possible.

Other examples surfaced during the audio-taped discussions following the problem-solving session. When asked if he knew what seams were in the problem Ronnie responded, "It means to have the seams to fit cause you got an extra board. If you have an extra board left over you can fill the seams with the extra boards." In contrast Melissa, who proved to be one of the better problem solvers responded to the same question with "It is the place where the boards meet." Even experience not directly related to the solving of the problem created difficulty. Christy had never experienced a situation where parts of a problem were presented on a new sheet of paper. She said, "I thought we were doing a whole different problem. If it were to be like (a) right under it, I could have understood."

Imposing unrequired restrictions. Frequently students would alter the goals of the problem statement by imposing restrictions that were not stated overtly but that they saw as implicit in the problem. For example, while trying to solve Part 1, Melissa states they need "six more without cutting." No mention was made in the problem that boards could not be cut.

In the same group Eric defends his position at one point by stating "He (John) might be looking for a better buy." This occurred during Part 2. No mention is made of cost until Part 4. Later in the same interaction Eric proposes some combination of 10-foot and 16-foot boards and claims John "wants to buy some of each." And, while solving Part 4, Patrick responds to a suggestion to purchase all 10-foot boards by saying "I don't think he would buy <u>all</u> 10-foot boards." All 10-foot boards would not have been the best suggestion, however, all 16-foot boards would have been close to a good approach. Patrick's imposing the restriction that John wouldn't buy all of one kind of board altered the way the group viewed the problem.

Use of metacognitive strategies. The majority of the students did not question their own thinking processes or strategies. In the situation of Christy's base of the room the group muddled their way toward a solution. One could conjecture that recent classroom experience of a formula for area (base times height) may have influenced her thinking. But from where did the ill-fated algorithm for finding the base of the room arise? Yet at no time did anyone ask, "What am I doing this for?" or "How will this help me?". Only two subjects, Damon and Melissa, showed evidence of metacognitive skills. The monitoring strategies of these two usually preceded a change in the representation of the problem. Although the modified representation was not always a correct representation, it often led to one.

For example, Melissa frequently reflected about her thinking. After carefully explaining to the other members of the group how she had arrived at a solution she said, "So, we still have four feet, and we have to figure out what we did wrong." After working a few more minutes she reaches the correct solution and states, "that solves the problem because that fills the entire room." Again later, on the fewest-seams part, she reaches an answer of 98 feet with one combination of boards but states, "But that doesn't solve our problem." And she hadn't! But with the help of another group member they modified their representation and moved closer to a reasonable solution.

The second subject to monitor his thinking was Damon in Group 1. On reaching an impasse in the solution process Damon would frequently utter, "Well, we have got to look this over a little bit more." "I was thinking we should . . ." "I was thinking if you . . ." Damon, however, was not as successful as Melissa in overall problem-solving success. He lacked basic computational skills and mathematical understandings.

Belief Systems. The beliefs or to use Schoenfeld's (1983) expression, the make sense epistemology the subject imposed on the problem situation frequently altered the representation they formed. Let's return to the different-problem-on-every-sheet-of paper situation. While this certainly could be attributed to lack of experience, it also could have developed from the overgeneralization of a narrow range of classroom experiences and the belief that new information occurs on new sheets of paper. Also, recall Patrick's imposing the idea that John wouldn't want to buy all of one kind of board. Underlying the assumption could certainly be the belief that in mathematical word problems you must use all the numbers.

The Group as a Facilitator

After identifying some of the factors that hindered successful problem solving for the individual (lack of experience, imposing unrequired restrictions, lack of monitoring strategies, and belief systems) the context of the group was analyzed. It was found that group interaction provided a useful framework within which individuals were able to form a clearer representation of the problem.

Collective Experience. The first factor of the group was that the collective often supplied the background information that individual students did not possess. For the subjects who did not understand what baseboard was, it was irrelevant to proceed with the problem and perform calculations that had no meaning. However, when members of the group had experience to share, the individuals in the group were able to create a more useful representation, as in the case of Darby saying "baseboards go around the room not in the room," in response to his group trying to calculate the area to ascertain the amount of baseboard that was needed in Part 2. The members of the group helped each other in building connections between elements of the problem and prior knowledge.

Group Monitoring. A second important factor of the group was that the challenge and disbelief of peers acted as a form of monitoring when self-monitoring was not apparent. Subjects seldom questioned their own strategies, but frequently challenged each other. Such an encounter forced students to examine their own strategies and beliefs more closely. For example, within a few seconds after reading the problem Leigh arrived at an answer of 588 for the area of the room in Part 1. Both Christy and Darby challenged her approach asking "Why?" Leigh ignored their comments so they continued to calculate the perimeter. Nearly two minutes later Leigh raised her head and said "I got 98 for the perimeter if I add." One can only surmise that she had silently rethought her approach for solving the problem. Darby and Christy had challenged her and she was forced to rethink her representation of the problem. Left alone she might have been satisfied with her thinking and followed a useless solution path.

<u>Format of Group Episodes</u>. The third factor of the group, which could be viewed as helpful in the problem-solving situation, was that the form the group sessions took was quite different from traditional classroom episodes. First, members seldom agreed upon an answer immediately. Similar to Noddings (1984) students, when a correct answer was proposed and no alternative answers were suggested, students took several minutes to discuss it. It provided the students the time to think that is so often neglected in the actual classroom setting. When groups moved too fast for an individual member, calls to "wait" would occur or, as in the case of Leigh, the individual would reflect internally and return to the group discussion when they were comfortable with the previous information.

Second, in contrast to the question, response, praise format commonly employed in the classroom, the group engaged in much more extended evaluation episodes. To agree on an answer after it had been suggested would often take several minutes even if no challenge were being offered. They seemed to need time to think about what was being suggested.

Hypotheses

Experience within and outside of the classroom affects not only problem-solving ability but the belief systems that influence problem-solving stratgies. A student's environment fosters beliefs that influence some of the decisions that are made in a problem-solving situation. A student who enters a problem-solving session without knowledge of baseboards may not have the cognitive structures necessary to create an accurate representation of the problem. The student who believes that all the numbers must be used in a word problem may employ different strategies than one who does not.

The small group may be useful in two aspects of problem-solving research. First, it provides a way to externalize internal dialogue and second, it may serve as a useful pedagogical format.

Two directions for future research are suggested by the results. First, a study using the small group as a pedagogical approach for improving problem-solving ability seems warranted. Second, a closer look at the factors that impede creation of an accurate representation is needed. The influence of metacognitive skills and belief systems are both factors that need investigation.

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